# STAT 593
# Duality / Conjugacy

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# Outline

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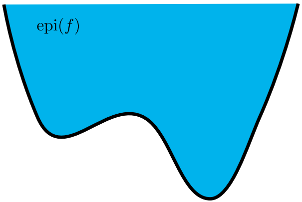
# Graphs and epigraphs

Notation : $\operatorname{dom} f = \{x \in \mathbb{R}^d : f(x) < +\infty\}$, $\overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$

**Definition**

For a function $f : \mathbb{R}^d \to \overline{\mathbb{R}}$, the **epigraph** of $f$ is the set

$$\operatorname{epi}(f) = \{(x,t) \in \mathbb{R}^d \times \overline{\mathbb{R}} : f(x) \leq t\}$$



# Convex function

A function $f : \mathbb{R}^d \to \overline{\mathbb{R}}$ is **convex** if for all $x, y \in \mathbb{R}^d$ and $\alpha \in [0, 1]$

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha) f(y)$$

**Proposition**

The function $f$ is convex iff $\mathrm{epi}(f)$ is a convex set.

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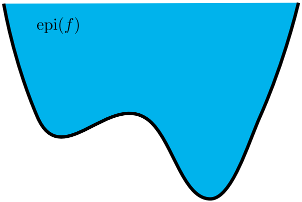


Non-convex

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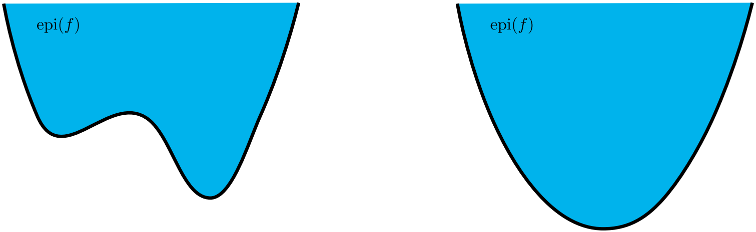


Non-convex Convex

# Convex hull of a function

**Definition**

The **convex hull** of a function $f : \mathbb{R}^d \to \overline{\mathbb{R}}$ is the function $\mathrm{conv}(f)$ whose epigraph is the convex hull of the epigraph of $f$ :

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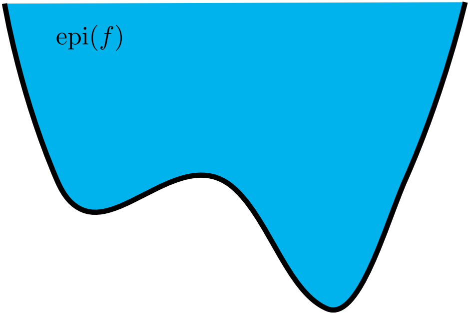

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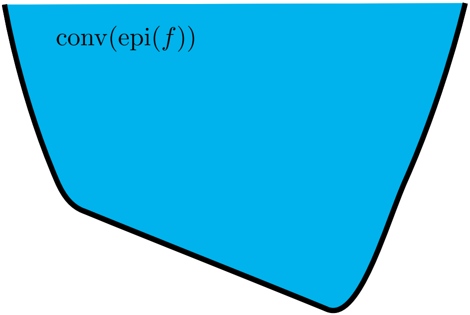

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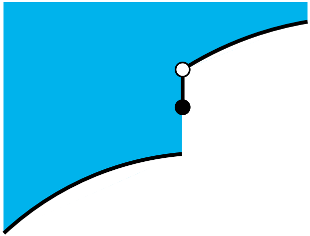

Non-closed function

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Non-closed function     Closed function

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The **closure** $\mathrm{cl}(f)$ of the function $f$ is defined as the function having for epigraph the closure of the epigraph of $f$ :

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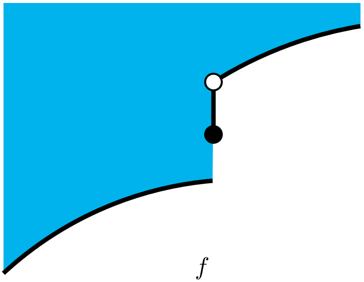

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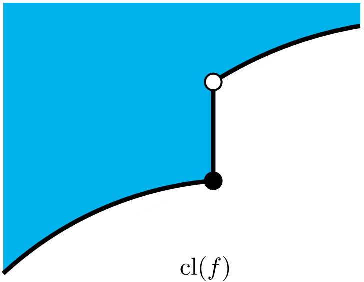

# Why closed functions ?

Motivation : close function convenient for minimization

- a closed function, defined on a nonempty closed and bounded set, is bounded below and attains its infimum

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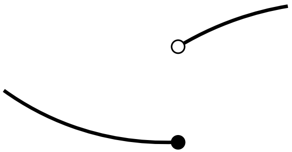

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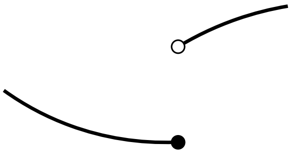

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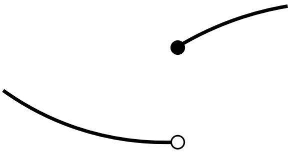

Non closed function : inf is not reached

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# Convex conjugate

Assumption : $f : \mathbb{R}^d \to \overline{\mathbb{R}}$, satisfies $f \not\equiv +\infty$ and there exists an affine function function minorizing $f$ on $\mathbb{R}^d$

**Definition**

The convex **conjugate** (a.k.a. the **Legendre-Fenchel transform**) of $f$ is the function $f^*$ defined by

$$\begin{aligned} f^* : \mathbb{R}^d &\to \overline{\mathbb{R}} \\ s &\mapsto \sup \{ \langle s, x \rangle - f(x) \ : \ x \in \operatorname{dom} f \} \end{aligned}$$

Rem: equivalently $\mapsto f^*(s) = -\inf \{ f(x) - \langle s, x \rangle \ : \ x \in \operatorname{dom} f \}$

Rem: $f^*$ is convex as a $\sup$ of convex (affine) functions

https://people.ok.ubc.ca/bauschke/Research/68.pdf

# Intuition

$$\boxed{f^*(s) = \sup\{\langle\, s\,,\, x\,\rangle - f(x)\ :\ x \in \operatorname{dom} f\}}$$

Interpretation : in the smooth case and when $\nabla f$ is one-one and $\operatorname{dom} f = \mathbb{R}^d$, let us define :

$$x(s) = \arg\max_{x \in \mathbb{R}^d} \langle\, s\,,\, x\,\rangle - f(x)$$

Then,

$$\begin{aligned} \nabla f(x(s)) = s \iff & (\nabla f)^{-1}(s) = x(s) \\ \iff & \nabla(f^*)(s) = x(s) \end{aligned}$$

All in all :

$$\nabla(f^*) = (\nabla f)^{-1}$$

[1]

1. J.-B. Hiriart-Urruty et C. Lemaréchal. *Convex analysis and minimization algorithms. II*. T. 306. Berlin : Springer-Verlag, 1993, Chapter X.

# Intermission

Movies on conjugacy insights (see associated notebook)

# First properties

Remind : $f^*(s) = \sup \{ \langle s , x \rangle - f(x) \ : \ x \in \mathrm{dom}\, f \}$

**Fenchel's inequality** :

$$\forall (x, s) \in \mathbb{R}^d \times \mathbb{R}^d, \quad f(x) + f^*(s) \geq \langle x , s \rangle$$

**Fenchel's equality** : equality in the former is equivalent to

$$s \in \partial f(x) \iff x \in \arg\max_{y \in \mathbb{R}^d} \langle s, y \rangle - f(y) \qquad \text{(Fermat's rule w.r.t. } x\text{)}$$

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# Bi-conjugate / closed convex hull

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\begin{aligned}
f^{**}(x) &= \sup_s \{ \langle s , x \rangle - f^*(s) \} \\
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\end{aligned}$$

Hence : $f^{**}$ is the **closed convex hull** of $f$ (whose epigraph is obtained as the intersection of all supporting half spaces)

# Standard functions

- $\boxed{f = \frac{1}{2}\left\|\cdot\right\|_2^2 \Rightarrow f^* = \frac{1}{2}\left\|\cdot\right\|_2^2}$; it is the only[2] $f$ s.t. $f^* = f$

2. J.-J. Moreau. "Proximité et dualité dans un espace hilbertien". In : *Bull. Soc. Math. France* 93 (1965), p. 273–299.

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- Let $p, q > 0$ s.t. $\frac{1}{p} + \frac{1}{q} = 1$, then $\boxed{f = \frac{1}{p}\left\|\cdot\right\|_p^p \Rightarrow f^* = \frac{1}{q}\left\|\cdot\right\|_q^q}$

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- Let $\iota_{\mathcal{C}}(x) = \begin{cases} 0 & \text{if } x \in \mathcal{C} \\ +\infty & \text{if } x \notin \mathcal{C} \end{cases}$

  Constant case : $\forall c \in \mathbb{R}^d$, $\boxed{f \equiv c \Rightarrow f^* = \iota_{\{-c\}}}$

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- Let $\mathcal{B}_{\|\cdot\|}(0,1)$ be a unit ball associated to a norm $\left\|\cdot\right\|$ and $\left\|\cdot\right\|_*$ is the dual norm associated to it where $\left\|x\right\|_* = \sup_{y:\|y\|\leq 1} \langle\, x \,,\, y \,\rangle$.

  Then, $\boxed{f = \iota_{\mathcal{B}_{\|\cdot\|}(0,1)} \Rightarrow f^* = \left\|\cdot\right\|_*}$

2. J.-J. Moreau. "Proximité et dualité dans un espace hilbertien". In : *Bull. Soc. Math. France* 93 (1965), p. 273–299.

# Simple properties

$$\forall \alpha > 0, \qquad (\alpha f)^* = \alpha f^*\left(\frac{\cdot}{\alpha}\right) \qquad \text{(Scaling)}$$

$$\forall \alpha > 0, \qquad \left(\alpha f\left(\frac{\cdot}{\alpha}\right)\right)^* = \alpha f^* \qquad \text{(Scaling)}$$

$$\forall \delta x \in \mathbb{R}^d, \qquad \left(f(\cdot - \delta x)\right)^* = f^* + \langle\, \delta x \,,\, \cdot \,\rangle \qquad \text{(Shifting)}$$

$$\forall \tau \in \mathbb{R}, \qquad (f + \tau)^* = f^* - \tau \qquad \text{(Shifting)}$$

$$\forall \delta x \in \mathbb{R}^d, \qquad (f + \langle\, \delta x \,,\, \cdot \,\rangle\,)^* = f^*(\cdot - \delta x) \qquad \text{(Shifting)}$$

$$\forall L \in \mathbb{R}^{d \times d}, \qquad (f \circ L)^* = f^* \circ (L^{-1})^\top \qquad \text{(Linear composition)}$$

(when $L$ invertible)

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# Fenchel duality [3]

Optimization : $\arg\min_{x \in \mathbb{R}^d} \mathcal{P}(x)$ where $\mathcal{P}(x) = f(Lx) + g(x)$

- $f : \mathbb{R}^n \to \mathbb{R}$ closed convex
- $g : \mathbb{R}^d \to \mathbb{R}$ closed convex
- $L : n \times d$ matrix

**Theorem**

$$\sup_s \{-f^*(s) - g^*(-L^\top s)\} \leq \inf_x \{f(Lx) + g(x)\}$$

Moreover, under mild assumptions, equality holds (**strong duality**)

3. H. H. Bauschke et P. L. Combettes. *Convex analysis and monotone operator theory in Hilbert spaces*. New York : Springer, 2011, p. xvi+468.

# Duality gap / vocabulary

**Primal** function : $\mathcal{P}(x) = f(Lx) + g(x)$ (to **minimize**)

Primal solution : $x^\star \in \arg\min_x \mathcal{P}(x)$

**Dual** function : $\mathcal{D}(s) = -f^*(s) - g^*(-L^\top s)$ (to **maximize**)

Dual solution : $s^\star \in \arg\min_s \mathcal{D}(s)$

**Duality gap** : $\Delta(x, s) = \mathcal{P}(x) - \mathcal{D}(s)$

**Theorem**

$$\forall (x, s), \quad 0 \leq \mathcal{P}(x) - \mathcal{P}(x^\star) \leq \Delta(x, s)$$

Also $\Delta(x, s) = 0$ implies that $\mathcal{P}(x) = \mathcal{P}(x^\star)$ and $\mathcal{P}(s) - \mathcal{P}(s^\star)$

# Duality gap for stopping algorithms

$$\Delta(x, s) = \mathcal{P}(x) - \mathcal{D}(s)$$

Stopping criterion for iterative solver :
Fix $\varepsilon > 0$ (small). Whenever one has point $x, s$ then

$$\boxed{\Delta(x, s) \leq \varepsilon \Rightarrow \mathcal{P}(x) - \mathcal{P}(x^\star) \leq \varepsilon}$$

Hence, stopping with duality gap criterion leads to a precise statement on the sub-optimality of the solution obtained.

Rem: this is a more precise criterion that choices like

$$\frac{\mathcal{P}(x^{t+1}) - \mathcal{P}(x_t)}{\mathcal{P}(x^t)} \leq \varepsilon \quad \text{or} \quad \nabla\mathcal{P}(x^t) \leq \varepsilon$$

# Example : duality gap for the standard median

$$\mathrm{Med}_n(\mathbf{x}) \in \arg\min_{\theta\in\mathbb{R}} \|\theta\mathbf{1}_n - \mathbf{x}\|_1 = \mathcal{P}(\theta)$$

where $\mathbf{1}_n = (1,\dots,1)^\top \in \mathbb{R}^n$

$$f = \|\cdot - \mathbf{x}\|_1, \qquad g = 0$$

$$f^* = \iota_{\mathcal{B}_{\|\cdot\|_\infty}(0,1)} + \langle \mathbf{x}, \cdot \rangle, \qquad g^* = \iota_{\{0\}}$$

Hence, $\mathcal{D}(s) = \langle s, \mathbf{x} \rangle + \iota_{\{s\in\mathbb{R}^n : \|s\|_\infty \leq 1\}} + \iota_{\{s\in\mathbb{R}^n : \mathbf{1}_n^\top s = 0\}}$

Rem: with iterate $\theta^t$ aiming at solving the primal problem, following would create dual feasible points :

$$s^t = \frac{\theta^t\mathbf{1}_n - \mathbf{x} - \mathrm{Ave}(\theta^t\mathbf{1}_n - \mathbf{x})}{\|\theta^t\mathbf{1}_n - \mathbf{x} - \mathrm{Ave}(\theta^t\mathbf{1}_n - \mathbf{x})\|_\infty}$$

# Example : duality gap for Lasso

Lasso objective : $\boxed{\mathcal{P}(\theta) = \frac{1}{2}\|X\theta - y\|_2^2 + \lambda\|\theta\|_1}$

- $f(z) = \frac{1}{2}\|z - y\|_2^2$; $f^*(s) = \frac{1}{2}\|s\|_2^2 + \langle s, y\rangle$ (use gradient)
- $g(\theta) = \lambda\|\theta\|_1$; $g^*(s) = \iota_{\{s, \|s\|_\infty \leq \lambda\}}$ ($\ell_\infty$ ball indicator)

- Duality gap :

$$\begin{aligned}\Delta(\theta, s) =& \mathcal{P}(\theta) + f^*(s) + g^*(-X^\top s) \\ =& \mathcal{P}(\theta) + \frac{1}{2}\|s\|_2^2 + \langle s, y\rangle\end{aligned}$$

as soon as $\|X^\top s\|_\infty \leq \lambda$, o.w. the bound is $+\infty$ (useless)

# More references

Material mostly inspired by the lecture notes by Pontus Giselsson :
http://www.control.lth.se/ls-convex-2015/

# References I


- BAUSCHKE, H. H. et P. L. COMBETTES. *Convex analysis and monotone operator theory in Hilbert spaces*. New York : Springer, 2011, p. xvi+468.
- HIRIART-URRUTY, J.-B. et C. LEMARÉCHAL. *Convex analysis and minimization algorithms. II*. T. 306. Berlin : Springer-Verlag, 1993.
- MOREAU, J.-J. "Proximité et dualité dans un espace hilbertien". In : *Bull. Soc. Math. France* 93 (1965), p. 273–299.