STAT 593
Robust statistics:
L-statistics: Linear combinations of order statistics

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Outline

L-estimates

Influence and robustness

Trimmed mean

Definition

Trimmed mean (at level α) : $\overline{x}_{n,\alpha} = \frac{1}{n-2m} \sum_{i=m+1}^{n-m} x_{(i)}$

where $m = \lfloor (n-1)\alpha \rfloor$ and $x_{(i)}$ denotes the order statistics **order statistics** $x_{(1)} \leq x_{(2)} \leq \ldots \leq x_{(n)}$

Rem: $\lfloor u \rfloor$ is the integer part of u

L-estimates

Definition

L-estimators are or the form

$$T_n(x_1, \ldots, x_n) = \sum_{i=1}^{n} a_i h(x_{(i)})$$

where the a_i are some coefficients ant the $x_{(i)}$ denote the order statistics **order statistics** $x_{(1)} \leq x_{(2)} \leq \ldots \leq x_{(n)}$

Often the weights are generated by choosing the

$$a_i = \int_{\frac{i-1}{n}}^{\frac{i}{n}} \lambda(x)dx$$

where $\lambda : [0, 1] \to \mathbb{R}_+$ satisfies $\int_0^1 \lambda(x)dx = 1$

Examples

$$\sum_{i=1}^{n} a_i h(x_{(i)}), \text{ with } a_i = \int_{\frac{i-1}{n}}^{\frac{i}{n}} \lambda(x)dx$$

- the α-Trimmed mean is recovered by choosing $h = \mathrm{Id}$ and $\lambda = \frac{1}{1-2\alpha}\mathbb{1}_{[\alpha,1-\alpha]}$
- the median is recovered by choosing $h = \mathrm{Id}$ and $\lambda = \delta_{\frac{1}{2}}$

Statistics / empirical c.d.f.

Let $F_n(x) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}_{[x_i,+\infty[}(x)$ the c.d.f. associated with the measure $\frac{1}{n}\sum_{i=1}^{n} \delta_{x_i}$ where δ_{x_i} are Dirac measures.

Then, one can write any statistic as $T_n(x_1, \ldots, x_n) = T(F_n)$

Rem: one refers to the property $\lim_{n\to\infty} T(F_n) = T(F)$, for *i.i.d.* observations $x_1, \ldots, x_n$ with distribution F, as Fisher consistency.

Examples

- For the mean $T(F) = \int x dF(x)$

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- For the median $T(F) = F^{-1}(\frac{1}{2})$
- For α-quantile $T(F) = F^{-1}(\alpha)$

Examples

- For the mean $T(F) = \int x dF(x)$
- For the median $T(F) = F^{-1}(\frac{1}{2})$
- For α-quantile $T(F) = F^{-1}(\alpha)$
- For an L-estimate with $\sum_{i=1}^{n} a_i h(x_{(i)})$ with $a_i = \int_{\frac{i-1}{n}}^{\frac{i}{n}} \lambda(x)dx$,
  $T(F) = \int h(F^{-1}(s))\lambda(s)ds = \int h(x)\lambda(F(x))dF(x)$,
  where $F^{-1}(s) = \inf \{x : F(x) \geq s\}$

Example continued

For the α-trimmed mean

$$T(F) = \frac{1}{1-2\alpha}\mathbb{E}_F[X\mathbb{1}_{[\alpha,1-\alpha]}(F(X))] = \frac{1}{1-2\alpha}\int_{\alpha}^{1-\alpha} F^{-1}(t)dt$$

Note that it can also be also be expressed as a (location)

M-estimate with $\psi(x) = \begin{cases} 0 & \text{for } x < F^{-1}(\alpha) \\ \frac{x}{1-2\alpha} & \text{for } F^{-1}(\alpha) \leq x \leq F^{-1}(1-\alpha) \\ 0 & \text{for } x > F^{-1}(1-\alpha) \end{cases}$

Example continued

For the α-trimmed mean

$$T(F) = \frac{1}{1-2\alpha}\mathbb{E}_F[X\mathbb{1}_{[\alpha,1-\alpha]}(F(X))] = \frac{1}{1-2\alpha}\int_\alpha^{1-\alpha} F^{-1}(t)dt$$

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Rem: not to be confused with the Windsor mean that uses

$$\psi(x) = \begin{cases} -\alpha & \text{for } x < F^{-1}(\alpha) \\ x & \text{for } F^{-1}(\alpha) \leq x \leq F^{-1}(1-\alpha) \\ \alpha & \text{for } x > F^{-1}(1-\alpha) \end{cases}$$

The later is associated to the statistic:

$$W(F) = (1-2\alpha)T(F) + \alpha F^{-1}(\alpha) + \alpha F^{-1}(1-\alpha)$$

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Influence function (directional derivative)

Definition

For a distribution F and a statistic T, the **influence function** of T at F is given for any x by

$$\begin{aligned} IF(x; F, T) &= \lim_{\epsilon \to 0} \frac{T[(1-\epsilon)F + \epsilon\delta_x] - T(F)}{\epsilon} \\ &= \lim_{\epsilon \to 0} \frac{T[F + \epsilon(\delta_x - F)] - T(F)}{\epsilon} \end{aligned}$$

Rem: if $\sup_x |IF(x, F, T)| = +\infty$, the influence of a single outlier might cause trouble; aim at $\sup_x |IF(x, F, T)| < +\infty$

Rem: the influence function is the directional derivative of $F \to T(F)$ taken in the direction of the Dirac function $\delta_x - F$, *i.e.*,

$$IF(x; F, T) = \nabla_{\delta_x - F} T(F)$$

Examples of influence functions

- For the mean $T(F) = \int t dF(t)$ so $IF(x, F, T) = x$, as

$$IF(x, F, T) = \lim_{\epsilon \to 0} \frac{(1-\epsilon)\int t dF(t) + \epsilon x - \int t dF(t)}{\epsilon} = x - T(F)$$

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General M-estimates

An M-estimate is a solution of $\hat{\theta}_n = \arg\min_{\theta \in \Theta} \sum_{i=1}^{n} \rho(x_i, \theta)$, or

equivalently for $\psi = \frac{\partial \rho}{\partial \theta}$: $\quad \sum_{i=1}^{n} \psi(x_i, \hat{\theta}_n) = 0$

Hence, $\hat{\theta}_n = T(F_n)$, where $T(F)$ is defined for any distribution F by $\int \psi(x, T(F)) dF(x) = 0$

- Location M-estimates:
  $\rho(x, \theta) = \rho(x - \theta)$ or equivalently $\psi(x, \theta) = \psi(x - \theta)$
- Scale M-estimates:
  $\rho(x, \theta) = \rho(x/\theta)$ or equivalently $\psi(x, \theta) = \psi(x/\theta)$

M-estimates and influence curve[1]

Theorem

For a regular M-estimate T and distribution F, the influence curve is given for any x_0 by:

$$IF(x_0, F, T) = \frac{-\psi(x_0, T(F))}{\int \frac{\partial \psi}{\partial \theta}\Big(x, T(F)\Big) dF(x)}$$

Rem: the regularity assumptions are left implicit here, and are only needed for interchanging integrals and limits

[1] F. R. Hampel et al. *Robust statistics: The Approach Based on Influence Functions*. Wiley series in probability and statistics. Wiley, 1986.

Proof

Fix x_0 and $\epsilon > 0$ and define $F_\epsilon = (1-\epsilon)F + \epsilon\delta_{x_0}$
Remind that $\int \psi(x, T(F_\epsilon))dF_\epsilon(x) = \int \psi(x, T(F))dF(x) = 0$, so

$$0 = \frac{1}{\epsilon}\left(\int \psi(x, T(F_\epsilon))dF_\epsilon(x) - \int \psi(x, T(F))dF(x)\right)$$

Proof

Fix x_0 and $\epsilon > 0$ and define $F_\epsilon = (1-\epsilon)F + \epsilon\delta_{x_0}$
Remind that $\int \psi(x, T(F_\epsilon))dF_\epsilon(x) = \int \psi(x, T(F))dF(x) = 0$, so

$$\begin{aligned}
0 =& \frac{1}{\epsilon}\left(\int \psi(x, T(F_\epsilon))dF_\epsilon(x) - \int \psi(x, T(F))dF(x)\right) \\
=& \frac{1-\epsilon}{\epsilon}\int \psi(x, T((1-\epsilon)F + \epsilon\delta_{x_0}))dF(x) \\
& + \psi(x_0, T((1-\epsilon)F + \epsilon\delta_{x_0}) - \frac{1}{\epsilon}\int \psi(x, T(F))dF(x)
\end{aligned}$$

Proof

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Hence, when $\epsilon \to 0$, providing T and ψ are continuous

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\end{aligned}$$

Hence, when $\epsilon \to 0$, providing T and ψ are continuous

$$-\psi(x_0, T(F)) = \lim_{\epsilon\to 0}\int \frac{\psi(x, T(F + \epsilon(\delta_{x_0} - F)) - \psi(x, T(F))}{\epsilon} dF(x)$$

Proof

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&+ \psi(x_0, T((1-\epsilon)F + \epsilon\delta_{x_0}) - \frac{1}{\epsilon}\int \psi(x, T(F))dF(x)
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$$\begin{aligned}
-\psi(x_0, T(F)) =& \lim_{\epsilon\to 0}\int \frac{\psi(x, T(F + \epsilon(\delta_{x_0} - F)) - \psi(x, T(F))}{\epsilon}dF(x) \\
=& \int \lim_{\epsilon\to 0}\frac{\psi(x, T(F + \epsilon(\delta_{x_0} - F)) - \psi(x, T(F))}{\epsilon}dF(x)
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$$\begin{aligned} 0 =& \frac{1}{\epsilon}\left(\int \psi(x, T(F_\epsilon))dF_\epsilon(x) - \int \psi(x, T(F))dF(x)\right) \\ =& \frac{1-\epsilon}{\epsilon}\int \psi(x, T((1-\epsilon)F + \epsilon\delta_{x_0}))dF(x) \\ &+ \psi(x_0, T((1-\epsilon)F + \epsilon\delta_{x_0}) - \frac{1}{\epsilon}\int \psi(x, T(F))dF(x) \end{aligned}$$

Hence, when $\epsilon \to 0$, providing T and ψ are continuous

$$\begin{aligned} -\psi(x_0, T(F)) &= \lim_{\epsilon\to 0}\int \frac{\psi(x, T(F+\epsilon(\delta_{x_0} - F)) - \psi(x, T(F))}{\epsilon}dF(x) \\ &= \int \lim_{\epsilon\to 0}\frac{\psi(x, T(F+\epsilon(\delta_{x_0} - F)) - \psi(x, T(F))}{\epsilon}dF(x) \\ &= \int \frac{\partial\psi}{\partial\theta}(x, T(F))dF(x) \cdot \underbrace{\lim_{\epsilon\to 0}\frac{T(F+\epsilon(\delta_{x_0} - F) - T(F)}{\epsilon}}_{IF(x_0, F, T)} \end{aligned}$$

Connections with M-estimation results[3]

It can be proved[2] the following result:

$$\sqrt{n}(\hat{\mu}_n - \breve{\mu}) \to_d \mathcal{N}(0, V^2), \text{where} \quad V^2 = \int IF(x, F, T)^2 dF(x)$$

$$\text{reminding} \quad IF(x_0, F, T) = \frac{-\psi(x_0, T(F))}{\int \frac{\partial \psi}{\partial \theta}\Big(x, T(F)\Big) dF(x)}$$

[2] D. D. Boos and R. J. Serfling. "A Note on Differentials and the CLT and LIL for Statistical Functions, with Application to M-Estimates". In: *Ann. Statist.* 8.3 (May 1980), pp. 618–624.

[3] F. R. Hampel et al. *Robust statistics: The Approach Based on Influence Functions*. Wiley series in probability and statistics. Wiley, 1986.

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- Location M-estimates: $\psi(x, \theta) = \psi(x - \theta)$ and we recover

$$V^2 = \frac{\int \psi^2(x) dF(x)}{(\int \psi'(x) dF(x))^2} \text{ for } T(F) = 0$$

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$$V^2 = \frac{\int \psi^2(x) dF(x)}{(\int \psi'(x) dF(x))^2} \text{ for } T(F) = 0$$

- Scale M-estimates: $\psi(x, \theta) = \psi(x/\theta)$ and we get
$$V^2 = \frac{\int \psi^2(x) dF(x)}{(\int x\psi'(x) dF(x))^2} \text{ for } T(F) = 1$$

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Other optimality point of view for robustness

- Result showing the ψ function one should use for location/scale models by minimizing the asymptotic variance under the constraint that the influence function is bounded p. 117, Hampel *et al.* (1986) (answer relies on "Huberization" of the score function)
- Connections between the optimal choice and for location/scale models and concomitant estimation is provided p. 172, Huber (1981)

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Median and influence curve

Proposition

Assume F has a p.d.f. f, then for the median $T(F) = F^{-1}(\frac{1}{2})$

$$IF(x, F, T) = \frac{1}{2f(F^{-1}(\frac{1}{2}))} \operatorname{sign}\left[x - F^{-1}(\tfrac{1}{2})\right]$$

Rem: for centered distributions one has $F^{-1}(\frac{1}{2}) = 0$ and

$$IF(x_0, F, T) = \frac{1}{2f(0)} \operatorname{sign}[x]$$

Rem: similar computation can be performed for any quantile $F^{-1}(s)$

Proof

For any $\epsilon > 0$, $F_\epsilon = (1-\epsilon)F + \epsilon\delta_x$ and $F_\epsilon \circ F_\epsilon^{-1}(\frac{1}{2}) = \frac{1}{2}$, so:

$$0 = \frac{F_\epsilon \circ F_\epsilon^{-1}(\frac{1}{2}) - F \circ F^{-1}(\frac{1}{2})}{\epsilon}$$

Proof

For any $\epsilon > 0$, $F_\epsilon = (1-\epsilon)F + \epsilon\delta_x$ and $F_\epsilon \circ F_\epsilon^{-1}(\frac{1}{2}) = \frac{1}{2}$, so:

$$0 = \frac{F_\epsilon \circ F_\epsilon^{-1}(\frac{1}{2}) - F \circ F^{-1}(\frac{1}{2})}{\epsilon}$$

$$0 = \frac{\left[F_\epsilon \circ F_\epsilon^{-1}(\frac{1}{2}) - F \circ F_\epsilon^{-1}(\frac{1}{2})\right] + \left[F \circ F_\epsilon^{-1}(\frac{1}{2}) - F \circ F^{-1}(\frac{1}{2})\right]}{\epsilon}$$

Proof

For any $\epsilon > 0$, $F_\epsilon = (1-\epsilon)F + \epsilon\delta_x$ and $F_\epsilon \circ F_\epsilon^{-1}(\frac{1}{2}) = \frac{1}{2}$, so:

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For the first term : $(\delta_x - F) \circ (F^{-1}(\frac{1}{2})) = -\frac{1}{2}\,\mathrm{sign}(x - F^{-1}(\frac{1}{2}))$, where we use the "abuse of notation" $\delta_x(t) = \mathbb{1}_{[x,\infty[}(t)$

Proof

For any $\epsilon > 0$, $F_\epsilon = (1-\epsilon)F + \epsilon\delta_x$ and $F_\epsilon \circ F_\epsilon^{-1}(\frac{1}{2}) = \frac{1}{2}$, so:

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For the first term : $(\delta_x - F) \circ (F^{-1}(\frac{1}{2})) = -\frac{1}{2}\,\mathrm{sign}(x - F^{-1}(\frac{1}{2}))$,
where we use the "abuse of notation" $\delta_x(t) = \mathbb{1}_{[x,\infty[}(t)$
For the second:

$$\lim_{\epsilon \to 0} \frac{F \circ F_\epsilon^{-1}(\frac{1}{2}) - F \circ F^{-1}(\frac{1}{2})}{\epsilon} = f \circ (F^{-1}(\tfrac{1}{2})) \cdot IF\left(x, F, F^{-1}(\tfrac{1}{2})\right)$$

Proof

For any $\epsilon > 0$, $F_\epsilon = (1-\epsilon)F + \epsilon\delta_x$ and $F_\epsilon \circ F_\epsilon^{-1}(\frac{1}{2}) = \frac{1}{2}$, so:

$$0 = \frac{F_\epsilon \circ F_\epsilon^{-1}(\frac{1}{2}) - F \circ F^{-1}(\frac{1}{2})}{\epsilon}$$

$$0 = \frac{\left[F_\epsilon \circ F_\epsilon^{-1}(\frac{1}{2}) - F \circ F_\epsilon^{-1}(\frac{1}{2})\right] + \left[F \circ F_\epsilon^{-1}(\frac{1}{2}) - F \circ F^{-1}(\frac{1}{2})\right]}{\epsilon}$$

For the first term : $(\delta_x - F) \circ (F^{-1}(\frac{1}{2})) = -\frac{1}{2}\,\mathrm{sign}(x - F^{-1}(\frac{1}{2}))$, where we use the "abuse of notation" $\delta_x(t) = \mathbb{1}_{[x,\infty[}(t)$

For the second:

$$\lim_{\epsilon \to 0} \frac{F \circ F_\epsilon^{-1}(\frac{1}{2}) - F \circ F^{-1}(\frac{1}{2})}{\epsilon} = f \circ (F^{-1}(\tfrac{1}{2})) \cdot IF\left(x, F, F^{-1}(\tfrac{1}{2})\right)$$

Hence,

$$IF(x, F, T) = \frac{1}{2f(F^{-1}(\frac{1}{2}))}\,\mathrm{sign}\left[x - F^{-1}(\tfrac{1}{2})\right]$$

L-estimates and influence curve[4]

For an L-estimate with $\sum_{i=1}^{n} a_i h(x_{(i)})$ with $a_i = \int_{\frac{i-1}{n}}^{\frac{i}{n}} \lambda(x)dx$,
$T(F) = \int h(F^{-1}(s))\lambda(s)ds = \int h(x)\lambda(F(x))dF(x)$,
where $F^{-1}(s) = \inf \{x : F(x) \geq s\}$

Proposition

Assume that F has a p.d.f. f, then

$$IF(x_0, F, T) = \int_0^1 \frac{sh'(F^{-1}(s))}{f(F^{-1}(s))}\lambda(s)ds - \int_{F(x)}^1 \frac{h'(F^{-1}(s))}{f(F^{-1}(s))}\lambda(s)ds$$

Proof: see p.56 Huber (1981)

[4] F. R. Hampel et al. *Robust statistics: The Approach Based on Influence Functions*. Wiley series in probability and statistics. Wiley, 1986.

Influence function for the trimmed mean

Remind that $T(F) = \frac{1}{1-2\alpha} \int_{\alpha}^{1-\alpha} F^{-1}(s)ds$ for $\alpha \in [0, \frac{1}{2}]$.

$$IF(x, F, T) = \begin{cases} \frac{F^{-1}(\alpha) - W(F)}{1-2\alpha} & \text{for } x < F^{-1}(\alpha) \\ \frac{x - W(F)}{1-2\alpha} & \text{for } F^{-1}(\alpha) \leq x \leq F^{-1}(1-\alpha) \\ \frac{F^{-1}(1-\alpha) - W(F)}{1-2\alpha} & \text{for } x > F^{-1}(1-\alpha) \end{cases}$$

where $W(F) = (1-2\alpha)T(F) + \alpha F^{-1}(\alpha) + \alpha F^{-1}(1-\alpha)$

Rem: when F is symmetric this simplifies to

$$IF(x, F, T) = \begin{cases} \frac{F^{-1}(\alpha)}{1-2\alpha} & \text{for } x < F^{-1}(\alpha) \\ \frac{x}{1-2\alpha} & \text{for } F^{-1}(\alpha) \leq x \leq F^{-1}(1-\alpha) \\ \frac{F^{-1}(1-\alpha)}{1-2\alpha} & \text{for } x > F^{-1}(1-\alpha) \end{cases}$$

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Sensitivity curves: the population version counterpart of influence curves

Definition

For a function T defining a statistics $T_n(x_1, \dots, x_n) = T(F_n)$ the **sensitivity curve** of is given for any x by

$$SC_n(x; T_n) = \frac{T(\frac{n-1}{n} F_{n-1} + \frac{1}{n}\delta_x) - T(F_{n-1})}{\frac{1}{n}}$$
$$= n[T_n(x_1, \dots, x_{n-1}, x) - [T_{n-1}(x_1, \dots, x_{n-1})]$$

Interpretation: effect of adding one datapoint on a given statistic:

$$T_n(x_1, \dots, x_{n-1}, x_n) = T_{n-1}(x_1, \dots, x_{n-1}) + \frac{1}{n} SC_n(x_n; T_n)$$

Rem: if $SC_n(x; T_n)$ is unbounded w.r.t. x, then the breakdown point of T_n is $\frac{1}{n+1}$

Examples

$$SC_n(x;T_n) = n[T_n(x_1,\ldots,x_{n-1},x) - [T_{n-1}(x_1,\ldots,x_{n-1})]$$

- Mean case: $SC_n(x;T) = n[\frac{n-1}{n}\overline{X}_{n-1} + \frac{x}{n} - \overline{X}_{n-1}] = x - \overline{X}_n$

References I

- Boos, D. D. and R. J. Serfling. "A Note on Differentials and the CLT and LIL for Statistical Functions, with Application to M-Estimates". In: *Ann. Statist.* 8.3 (May 1980), pp. 618–624.
- Hampel, F. R. et al. *Robust statistics: The Approach Based on Influence Functions*. Wiley series in probability and statistics. Wiley, 1986.
- Huber, P. J. *Robust Statistics*. John Wiley & Sons Inc., 1981.