# Stable Recovery with Analysis Decomposable Priors

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***Abstract*—In this paper, we investigate in a unified way the structural properties of solutions to inverse problems regularized by the generic class of semi-norms defined as a decomposable norm composed with a linear operator, the so-called analysis decomposable prior. This encompasses several well-known analysis-type regularizations such as the discrete total variation, analysis group-Lasso or the nuclear norm. Our main results establish sufficient conditions under which uniqueness and stability to a bounded noise of the regularized solution are guaranteed.**

## I. Introduction

**Problem statement** Suppose we observe

$$y = \Phi x_0 + w, \quad \text{where} \quad \|w\| \leqslant \varepsilon \ ,$$

where $\Phi$ is a linear operator from $\mathbb{R}^N$ to $\mathbb{R}^M$ that may have a non-trivial kernel. We want to robustly recover an approximation of $x_0$ by solving the optimization problem

$$x^\star \in \underset{x \in \mathbb{R}^N}{\operatorname{Argmin}} \ \tfrac{1}{2}\|y - \Phi x\|^2 + \lambda R(x) \ , \ \text{where } R(x) := \|L^* x\|_{\mathcal{A}} \ , \quad (1)$$

with $L : \mathbb{R}^P \to \mathbb{R}^N$ a linear operator, and $\| \cdot \|_{\mathcal{A}} : \mathbb{R}^P \to \mathbb{R}^+$ is a decomposable norm in the sense of [**?**]. Decomposable regularizers are intended to promote solutions conforming to some notion of simplicity/low complexity that complies with that of $L^* x_0$. This motivates the following definition of these norms.

**Definition 1.** *A norm* $\| \cdot \|_{\mathcal{A}}$ *is* decomposable *at* $\beta \in \mathbb{R}^P$ *if there is a subspace* $T \subseteq \mathbb{R}^P$ *and a vector* $e \in T$ *such that*

$$\partial \| \cdot \|_{\mathcal{A}}(\beta) = \left\{ u \in \mathbb{R}^P \ : \ \mathcal{P}_T(u) = e \quad \textit{and} \quad \|\mathcal{P}_{T^\perp}(u)\|_{\mathcal{A}}^* \leqslant 1 \right\}$$

*and for any* $z \in T^\perp$, $\|z\|_{\mathcal{A}} = \sup_{v \in T^\perp, \|v\|_{\mathcal{A}}^* \leqslant 1} \langle v, \, z \rangle$, *where* $\| \cdot \|_{\mathcal{A}}^*$ *is the dual norm of* $\| \cdot \|_{\mathcal{A}}$, $\mathcal{P}_T$ *(resp.* $\mathcal{P}_{T^\perp}$*) is the orthogonal projector on* $T$ *(resp. on its orthogonal complement* $T^\perp$*).*

Popular examples covered by decomposable regularizers are the $\ell_1$-norm, the $\ell_1$-$\ell_2$ group sparsity norm, and the nuclear norm.

**Contributions and relation to prior work** In this paper, we give sufficient conditions under which (1) admits a unique minimizer. Then we develop results guaranteeing that a stable approximation of $x_0$ can be obtained from the noisy measurements $y$ by solving (1), with an $\ell_2$-error that comes within a factor of the noise level $\varepsilon$. This goes beyond [**?**] which considered identifiability in the noiseless case, with $L = \mathrm{Id}$ and $\Phi$ a Gaussian matrix. $\ell_2$-stability is also studied in [**?**] for $L = \mathrm{Id}$ under stronger sufficient assumptions than ours. Our results generalize the stability guarantee of [**?**] established when the decomposable norm is $\ell_1$ and $L$ is a frame. A general stability result for sublinear $R$ is given in [**?**]. The stability is however measured in terms of $R$, and $\ell_2$-stability can only be obtained if $R$ is coercive, *i.e.,* $L^*$ is injective.

## II. Uniqueness

We first note that traditional coercivity and convexity arguments allow to show that the set of (global) minimizers of (1) is a non-empty compact set if and only if $\ker(\Phi) \cap \ker(L^*) = \{0\}$.

We shall now give a sufficient condition under which problem (1) admits exactly one minimizer. The following assumptions will play a pivotal role in our analysis throughout the paper.

**Assumption** $(\mathrm{SC}_x)$ There exist $\eta \in \mathbb{R}^M$ and $\alpha \in \partial \| \cdot \|_{\mathcal{A}}(L^* x)$ such that the following so-called source condition is verified:

$$\Phi^* \eta = L \alpha \in \partial R(x) \ .$$

**Assumption** $(\mathrm{INJ}_T)$ Let $T$ be the subspace in Definition 1 associated to $L^* x$. $\Phi$ is injective on $\ker(\mathcal{P}_{T^\perp} L^*)$.

It is immediate to see that since $\ker(L^*) \subseteq \ker(\mathcal{P}_{T^\perp} L^*)$, $(\mathrm{INJ}_T)$ implies that the set of minimizers is indeed non-empty and compact.

**Theorem 1.** *For a minimizer* $x^\star$ *of* (1)*, let* $T_\star$ *and* $e_\star$ *be the subspace and vector in Definition 1 associated to* $L^* x^\star$*. Assume that* $(\mathrm{SC}_{x^\star})$ *is verified with* $\|\mathcal{P}_{T_\star^\perp}(\alpha)\|_{\mathcal{A}}^* < 1$*, and that* $(\mathrm{INJ}_{T_\star})$ *holds. Then,* $x^\star$ *is the unique minimizer of* (1)*.*

## III. Stability to noise

We are now ready to state our main stability result.

**Theorem 2.** *Let* $T$ *and* $e$ *be the subspace and vector in Definition 1 associated to* $L^* x_0$*. Assume that* $(\mathrm{SC}_{x_0})$ *is verified with* $\|\mathcal{P}_{T^\perp}(\alpha)\|_{\mathcal{A}}^* < 1$*, and that* $(\mathrm{INJ}_T)$ *holds. Then, for* $\lambda = c\varepsilon$

$$\|x^\star - x_0\| \leqslant C\varepsilon \ ,$$

*where* $C = C_1 \left(2 + c\|\eta\|\right) + C_2 \frac{(1 + c\|\eta\|/2)^2}{c(1 - \|\mathcal{P}_{T^\perp}(\alpha)\|_{\mathcal{A}}^*)}$*, and* $C_1 > 0$ *and* $C_2 < 0$ *are constants independent of* $\eta$ *and* $\alpha$*.*

In the following corollary, we provide a stronger sufficient stability condition. It will allow to construct good dual vectors $\eta$ and $\alpha$ that are computable, which in turn yield explicit constants in the bound. For this, suppose that $(\mathrm{INJ}_T)$ is verified, and define $\mathrm{IC}(T, e) = \min_{u \in \ker(L\mathcal{P}_{T^\perp})} \|\Gamma^{[T^\perp]} e + \mathcal{P}_{T^\perp} u\|_{\mathcal{A}}^*$ with $\Gamma^{[T^\perp]} = (L\mathcal{P}_{T^\perp})^+ (\Phi^* \Phi A^{[T^\perp]} - \mathrm{Id}) L \mathcal{P}_T$ and $A^{[T^\perp]} = U\left(U^* \Phi^* \Phi U\right)^{-1} U^*$, and $U$ is a matrix whose columns form a basis of $\ker(\mathcal{P}_{T^\perp} L^*)$. Note that $\mathrm{IC}(T, e)$ can be computed by solving a convex program. It also specializes to the criterion developed in [**?**] for the case of the $\ell_1$ analysis prior.

**Corollary 1.** *Assume that* $\mathrm{IC}(T, e) < 1$*. Then, taking* $\eta = \Phi A^{[T^\perp]} L \mathcal{P}_T e$*, there exists* $\alpha$ *such that* $(\mathrm{SC}_{x_0})$ *is satisfied. Moreover, the bound of Theorem 2 holds true substituting* $1 - \mathrm{IC}(T, e)$ *for* $1 - \|\mathcal{P}_{T^\perp}(\alpha)\|_{\mathcal{A}}^*$.